Ideal Bose gas

Again, there are $g \frac{4\pi p^2 dp}{(2\pi\hbar)^3} V$ states in a spherical layer of width dp

But now there may be >1 particles (bosons) in each state

$$\int g \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{1}{e^{\frac{\varepsilon(p)-\mu}{T}}-1} = \frac{N}{V}$$

Changing variables to $\varepsilon(p) = \frac{p^2}{2m}$,

$$\frac{N}{V} = \frac{g\, m^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3} \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\frac{\varepsilon-\mu}{T}}-1}$$

Note an important distinction between Fermi and Bose gases. At $T=0$ a Fermi gas has a finite, while a Bose gas may sit entirely in the state with $E=0$ at $T=0$

(Reminder : $E=0$ is the ground state, and all the bosons will want to be there)

Let's lower the temperature gradually. Will the state with all bosons in the ground state be reached at $T>0$? Assume so.

There are 2 possible scenarios

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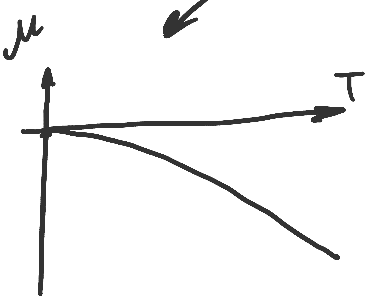
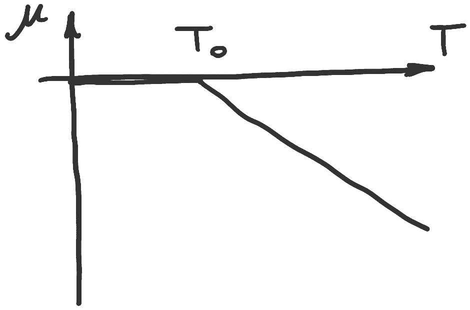
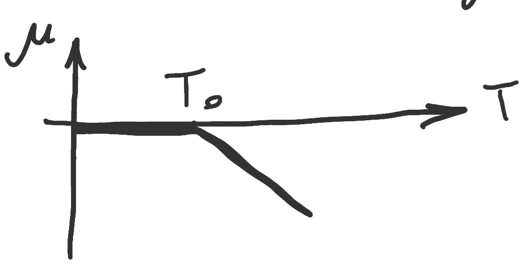




$$\frac{N}{V} = \frac{g\,(mT)^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3} \underbrace{\int_0^\infty \frac{\sqrt{z}\,dz}{e^z - 1}}_{\frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right)} \qquad \left(z = \frac{\varepsilon}{T}\right)$$

$$\frac{N}{V} = \frac{g\,(mT)^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}\hbar^3}\,\zeta\left(\frac{3}{2}\right)$$

$$T_0 = \frac{2\pi}{\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{2}{3}}} \frac{\hbar^2}{m g^{\frac{2}{3}}} \left(\frac{N}{V}\right)^{\frac{2}{3}} \approx \frac{3.31}{g^{\frac{2}{3}}} \frac{\hbar^2}{m} \left(\frac{N}{V}\right)^{\frac{2}{3}}$$



(What about the condition of having $\mu < 0$ for all temperatures in a Bose gas?)

One has to be more careful with the level $E = 0$. There is some uncertainty there. ... how many

One [illegible] the level $E=0$. There is some uncertainty
we cannot tell from this formula how many
particles there are on the level with $E=0$.

But let us count the particles with $E>0$

$$\frac{gV(mT)^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{\sqrt{z}\,dz}{e^z-1} = N\cdot\left(\frac{T}{T_0}\right)^{\frac{3}{2}}$$

(for $\mu=0$)

$$\boxed{N_{\varepsilon=0} = N\left[1-\left(\frac{T}{T_0}\right)^{\frac{3}{2}}\right]}$$

BEC is often related to superconductivity and superfluidity (e.g. He^4).
In 1995 observed explicitly by
E. Cornell and C. Wieman
// The most famous picture

Let's compute the energy at $T<T_0$

$$E = \frac{gV(mT)^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{\varepsilon^{\frac{3}{2}}\,d\varepsilon}{e^{\frac{\varepsilon}{T}}-1} =$$

$$= \frac{gV(mT)^{\frac{3}{2}}T}{2^{\frac{1}{2}}\pi^2\hbar^3}\underbrace{\int \frac{z^{\frac{3}{2}}\,dz}{e^z-1}}_{\shortparallel} = \frac{3g\,m^{\frac{3}{2}}T^{\frac{5}{2}}V}{2^{\frac{5}{2}}\pi^{\frac{3}{2}}\hbar^3}\zeta\left(\frac{5}{2}\right)$$

$$= \frac{3\sqrt{\pi}}{4}\,\zeta\left(\frac{5}{2}\right)$$

The specific heat is $C_V = \left(\frac{dE}{dT}\right)_V = \frac{5}{2}\frac{E}{T}$

Thus, $C_V \propto T^{\frac{3}{2}}$

$$\left.\begin{array}{l} C_V \propto T^{\frac{3}{2}} \\ C_V = T\left(\frac{dS}{dT}\right)_V \end{array}\right\} \rightarrow S = \frac{2}{3}\,\frac{5}{2}\,\frac{E}{T} = \frac{5}{3}E$$

$$F = E - TS = -\frac{2}{3}E$$

(Note that the number of particles is conserved)

So, $F = -\dfrac{g\, m^{\frac{3}{2}}\, T^{\frac{5}{2}}\, V}{2^{\frac{5}{2}}\,\pi^{\frac{3}{2}}\,\hbar^3}\,\zeta\left(\frac{5}{2}\right)$

The pressure

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{g\, m^{\frac{3}{2}}\,\zeta\left(\frac{5}{2}\right)}{2^{\frac{5}{2}}\,\pi^{\frac{3}{2}}\,\hbar^3}\; T^{\frac{5}{2}}$$